

Shell-crossings in Gravitational Collapse

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An important issue in the study of continual gravitational collapse of a massive matter cloud in general relativity is whether shell-crossing singularities develop as the collapse evolves. We examine this here to show that for any spherically symmetric collapse in general, till arbitrarily close to the final singularity, there is always a finite neighborhood of the center in which there are no shell-crossings taking place. It follows that in order to study the visibility or otherwise of the ultra-dense region close to the final singularity of collapse where physical radius of the matter cloud shrinks to an arbitrarily small value, we can always consider without loss of generality a collapsing ball of finite comoving radius in which there are no shell-crossings.

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Black holes have found many applications in modern astronomy and astrophysics today, especially for very high energy observed phenomena in the universe. Such black holes would be possibly created when large massive stars collapse under the pull of their own gravity on exhausting their internal nuclear fuel. One of the key important problem in this connection is that of black hole formation in gravitational collapse of a massive star. The general theory of relativity predicts that the final outcome of a continual collapse is a spacetime singularity, where the mass-energy densities, spacetime curvatures and other physical quantities blow up to become arbitrarily large. But general relativity does not predict that such a singularity will be necessarily covered in an event horizon, forming a black hole.

Therefore the question is, when a massive star collapses under its own gravity if it would necessarily form a black hole. What is needed to resolve this profound issue at the heart of modern black hole physics and its astrophysical applications is a careful study of the collapse phenomena within the framework of gravitation theory. Such a treatment of dynamical collapse would be essential to determine the final fate of a massive collapsing star which shrinks catastrophically under the force of its own gravity.

The continual gravitational collapse of a matter cloud in the framework of general relativity was first investigated by Oppenheimer and Snyder, and Datt [1,2].

They considered a homogeneous spherical star with vanishing internal pressures and zero rotation. It was shown that under these idealized conditions, the cloud collapses simultaneously to a spacetime singularity, covered within an event horizon that develops before the singularity as the collapse proceeds. Thus a black hole develops in the spacetime which hides the singularity from any faraway observers. This classic picture became the foundation of an extensive theory and astrophysical applications of modern black hole physics further to the cosmic censorship conjecture [3], namely that all realistic massive stars undergoing a continual gravitational collapse have the same qualitative behaviour, i.e. the spacetime singularity of collapse must be always covered by an event horizon of gravity and hidden within a black hole.

While extensive theory and applications of black hole physics developed in past decades based on this assumption, the cosmic censorship conjecture remained unproved despite efforts of many decades. On the other hand, many collapse scenarios have been found in gravitation theory in past years, where a dynamically evolving gravitational collapse would terminate in a naked singularity without horizon, rather than a black hole. Therefore much effort has been devoted towards understanding and analyzing the final fate of physically realistic gravitational collapse scenarios to determine under what situations the collapse ends in a black hole and when a naked singularity will form, not covered by an event horizon in violation to the censorship conjecture (see e.g. Refs [4-14, 15] and references therein). The main point in these studies is to determine the nature of the spacetime singularity of collapse and the super ultra-dense regions near the same where the physical radius of the cloud goes to a vanishingly small value, in terms of its visibility or otherwise to external observers.

Towards such a purpose the dynamical collapse evolution of a massive matter cloud is to be studied and investigated using Einstein equations. In this connection, an important issue in the theory of gravitational collapse is that of formation of the so called ‘shell-crossing singularities’ that could develop, as opposed to the final genuine singularity where the entire cloud collapses to a zero physical radius. At a shell-cross, nearby shells of matter intersect creating momentary density singularities where some of the curvature scalars could blow up [16-19]. While these are known to be weak singularities which are resolved through a suitable extension of the spacetime (as opposed to big bang or strong curvature shell-focusing singularities), the coordinate system used to study the collapse evolution could break down at such shell-crossings, and so the conclusions on the

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final singularity become unclear in that case. Hence most of the gravitational collapse studies generally assume that there are no shell-crossings present within the evolving matter cloud.

Our purpose here is to show that in fact such an assumption is not needed, and that given any spherically symmetric collapse model, there always exists a finite neighborhood of the central line $r = 0$ such that there are no shell-crossings present in this ball of coordinate radius $[0, r_1]$, through out the evolution of collapse upto any arbitrarily close epoch to the final curvature singularity. Therefore, given any physically realistic collapsing star with a boundary at the coordinate radius $r = r_b$, if we are to study the visibility or otherwise of the super ultra-dense region very close to the central shell-focusing singularity at $r = 0$, we can always consider a neighborhood $[0, r_1]$ of the central line and examine the collapse evolution all the way without encountering any shell-crosses. Thus the essential nature of the final singularity can be determined without bothering for any shell-crosses in between.

We note that shell-crosses have been regarded generally as weak singularities (see e.g. Ref. 20 and the discussion therein). This is because, non-spacelike geodesics falling into a shell-crossing would not generally get focused into a surface or a line. Thus the volume elements along the geodesics are not crushed to a zero size, unlike at the big bang or a strong curvature shell-focusing singularity at the center of the Schwarzschild spacetime. In that sense, the material objects hitting a shell-crossing are not crushed out of existence, which is the signature of a genuine spacetime singularity. Also, in real astrophysical objects when densities are high then pressure gradients are present which are important and these may prevent the occurrence of shell-crossings. Actually, the shell-crossings seen sometimes in the Lemaitre-Tolman-Bondi models are not general enough, and it is believed that these could be a zero-pressure limit of an acoustic wave of high but finite density.

The shell-crossing singularities were studied also by Szekeres and Lun [21], who considered Newtonian and general relativistic spherically symmetric dust solutions and suggested the following criterion for a singularity to be classified as a shell-cross: (1) All Jacobi fields have finite limits (in an orthonormal parallel propagated frame) as they approach the singularity. (2) The boundary region forms an essential C^2 singularity which is C^1 regular, that is, it can be transformed away by a C^1 coordinate transformation. This allows one to think that such a shell-cross can be possibly avoided if the shapes of the arbitrary functions available in the geometry are properly chosen. Generally there are two ways suggested to avoid the shell-crossings. The first one is by setting up the functions so that the spatial derivative of the physical radius R' does not vanish throughout the collapsing cloud, and the second is by setting up the functions so that $R' = 0$ only at those locations $r = r_s$ where $M' = 0$, and $\lim |M'/R'|$ is finite, as r approaches r_s [22].

We show below that the first situation above is in fact always realized in a general spherically symmetric collapse, in a finite neighborhood of the central line, upto any epoch arbitrarily close to the final central singularity of collapse.

The general spherically symmetric line element describing the collapsing matter cloud can be written as,

$$ds^2 = -e^{2\nu(t,r)} dt^2 + e^{2\psi(t,r)} dr^2 + R(t,r)^2 d\Omega^2, \quad (1)$$

with the stress-energy tensor for a generic matter source given by,

$$T_t^t = -\rho; \quad T_r^r = p_r; \quad T_\theta^\theta = T_\phi^\phi = p_\theta. \quad (2)$$

The above is a general scenario, in that it involves no assumptions on the form of the matter or the equation of state. The dynamical evolution of the collapsing cloud and its final endstate is governed by the Einstein equations, which we study to understand the nature of the singularity of collapse. The visibility or otherwise of the singularity or the region close to it is determined by the behavior of the apparent horizon in the spacetime, which is the boundary of the trapped surface region that develops as the collapse progresses.

We define a scaling function $v(r, t)$ by the relation,

$$R = rv. \quad (3)$$

where R is the physical radius of the cloud. The Einstein equations are then written as,

$$p_r = -\frac{\dot{F}}{R^2 \dot{R}}, \quad (4)$$

$$\rho = \frac{F'}{R^2 R'}, \quad (5)$$

$$\nu' = 2 \frac{p_\theta - p_r}{\rho + p_r} \frac{R'}{R} - \frac{p_r'}{\rho + p_r}, \quad (6)$$

$$2\dot{R}' = R' \frac{\dot{G}}{G} + \dot{R} \frac{H'}{H}, \quad (7)$$

$$F = R(1 - G + H), \quad (8)$$

where the functions H and G are defined as,

$$H = e^{-2\nu(r,v)} \dot{R}^2, \quad G = e^{-2\psi(r,v)} R'^2. \quad (9)$$

In the above, we have five equations in seven unknowns, namely ρ , p_r , p_θ , R , F , G , H . Here ρ is the mass-energy density, p_r and p_θ are the radial and tangential pressures, R is the physical radius for the matter cloud, and F is the Misner-Sharp mass function. With the definitions (3) and (9), we can substitute the unknowns R , H with v , ν . Without loss of generality, the scaling function v can be set $v(t_i, r) = 1$ at the initial time $t_i = 0$ when the collapse commences. It then goes to zero at the final spacetime singularity t_s , which corresponds to $R = 0$, i.e. $v(t_s, r) = 0$. This amounts to the scaling $R = r$ at the initial epoch, which is an allowed freedom. The collapse condition here is the requirement that $\dot{R} < 0$ throughout the evolution, which is equivalent to $\dot{v} < 0$.

With this formalism for spherical collapse, we can consider now a continual collapse, with v evolving from $v = 1$ (initial epoch) to $v = \epsilon$, the later being an arbitrarily small positive quantity corresponding to the region of ultra-high density and pressure arbitrarily close to the final singularity epoch $v = 0$.

Then what we show below is: For any arbitrarily late stage of collapse, corresponding to $v = \epsilon$, where $\epsilon > 0$ is an arbitrarily small quantity, there always exists a δ such that in the $[0, r_1 = \delta]$ neighborhood of the central line $r = 0$, there are no shell-crossing singularities.

Thus the point is, given any general spherical collapse, there is always a value $r = r_1$, such that there are no shell-crosses in the cloud when the cloud boundary is $[0, r_1]$. Then we can consider the collapse all the way arbitrarily close to singularity to examine the visibility or otherwise of the arbitrarily small collapsed ball of ultra high densities and pressure without problem of shell-crossings developing in between. Since there is no scale in the problem, such a $[0, \delta]$ cloud for the range of comoving coordinate r is as good as any other finite cloud in principle. So we can choose the boundary of the compact collapsing object within this comoving radius r_1 , and for such a cloud there are no shell-crossings occurring so we can examine the nature of the singularity and the region very close to it in terms of its visibility or otherwise. Thus, given any arbitrary small $\epsilon > 0$, there are no shell-crossing singularities occurring in a finite neighborhood of the central line $r = 0$, that is, in a finite collapsing cloud for the entire evolution in the range $[1, \epsilon]$ for v . So we can regularly evolve the collapse from the regular initial surface $v = 1$ to any $v = \epsilon$, arbitrarily close to the final shell-focusing singularity, without any shell-crossing occurring in the cloud of a finite size around the central line.

We consider a compact collapsing matter cloud, which has the boundary $r = r_b$, i.e. the radial coordinate r is in the interval $[0, r_b]$. We consider a function J defined on the domain $D = [0, r_b] \times [\epsilon, 1]$ as $J(r, v) = v'$. Since the metric function $v(r, t)$ is C^2 at all regular spacetime points in the variables r and t , so $J(r, v)$ is a C^1 function of r and v , and hence a continuous function.

We note that with the scaling $R = r$ or $v = 1$ at $t = 0$, we have $R' = v + rv'$. The metric function $v(r, t)$ is C^2 or $J(r, v)$ is C^1 at all regular spacetime points. Firstly, note that at the initial surface we have $v = 1$ and the metric functions are C^2 . Also, at the central line $r = 0$, we have $R' = v + rv' = v$, and so R' is always positive and finite at the center, so no shell-cross singularity occurs at the central line of the cloud through out the collapse. Also, on any $v = \text{const.}$ surface, as we move away from the center, there is no shell-cross if $v' = 0$ or $v' > 0$. If $v' < 0$, even then there is no shell-cross on that surface till at least some finite value $r_1 > 0$, i.e. in the interval $[0, r_1]$. It follows that at the initial surface the metric functions are C^2 and continue to be so at the center line without any shell-crosses, and also in a certain neighborhood of the same as seen above, till the final shell focusing singularity is reached at $v = 0$.

Since the domain D is compact, J is bounded and hence there is a positive number M such that $|J(r, v)| \leq M$ for all (r, v) in D . In other words, M is the supremum of $|J(r, v)|$ taken over the domain D . We now show that whatever is the sign of v' , if we take $\delta = \frac{\epsilon}{M}$, then, for all r such that $0 < r < \delta$, the quantity R' is always positive. In the following, ϵ is to be taken small enough so that $\frac{\epsilon}{M} < r_b$, i.e. we remain within the cloud.

We consider now three cases as below:

Case 1: Firstly, we consider the case when $v' > 0$. In this case, obviously, since $v > \epsilon > 0$ and $r > 0$, we have the minimum of $R' = v + rv' > \epsilon > 0$, and so there are no shell-crossings throughout the collapse evolution in this case.

Case 2: Now suppose $v' < 0$, i.e. the function $J(r, v)$ is negative throughout the domain D . In this case, considering M and δ as above we see that for $0 < r < \delta$, we get (since $|v'| = -v'$ and since $|v'| < M$), $r < \frac{\epsilon}{M} < -\frac{v}{v'}$. Hence, $-rv' < v$ or $v + rv' > 0$. i.e. R' is positive throughout, thereby avoiding the shell-crossings again.

Case 3: Finally, it may be the case that the function $J(r, v)$ takes zero or positive values for some points (r, v) in the domain D , and takes negative values for remaining points in D . For those values of r and v for which $J(r, v)$ takes zero or positive values, R' is positive as shown in *Case 1*. Vanishing of $J(r, v)$ at any point means v is a positive constant for those values of r , which is the case, for example, at the initial epoch where $v = 1$.

In order to consider for the values (r, v) where $J(r, v) < 0$, let D_1 denote the subset of D which consists of all such (r, v) points. Now let M_1 be the supremum of $|J(r, v)|$ taken over the set D_1 . Then $M_1 \leq M$, and hence $\frac{1}{M} \leq \frac{1}{M_1}$. Thus, for those $(r, v) \in D$ for which $J(r, v) < 0$, and for all $r < \delta = \frac{\epsilon}{M}$, we get, $r < \frac{\epsilon}{M} \leq \frac{\epsilon}{M_1} < \frac{v}{|J(r, v)|} = \frac{v}{|v'|} = \frac{v}{-v'}$ on D_1 . This implies, as in the *Case 2*, that for $r < \delta$, we have $R' > 0$, and therefore the shell-crossings are avoided.

It follows that whatever is the behavior of $J(r, v)$, it is always possible to choose a neighborhood of the central line $r = 0$ in which shell-crossings can be avoided. One can also consider this in the following alternative way. Since the metric function R is a C^2 function, it follows that R' is a C^1 function of (r, t) or r and v , and hence is continuous in both r and v . Then we can give a continuity argument to ensure that we have R' positive in the neighborhood of $r = 0$. Along $r = 0$, as long as $v > 0$, clearly R' is positive. Hence, by continuity of R' , there will be a neighborhood of $r = 0$ where R' remains positive, irrespective of the sign of v' . What we have shown above is that a finite neighborhood or a finite radius ball around the central line $r = 0$ exists where there are no shell-crossings taking place.

We can consider, as an illustration, how the above works for the Lemaitre-Tolman-Bondi dust collapse models, which have been widely studied by many authors [23-28].

In this case, where we consider marginally bound collapse, the Einstein field equations above can be written down with $p_r = p_\theta = 0$ and these are solved to obtain,

$$R(r, t) = [\frac{9}{4}F(r)(t - a(r))^2]^{1/3}, \quad (10)$$

or,

$$t - a(r) = -\frac{2}{3} \frac{R^{3/2}}{\sqrt{F(r)}} \quad (11)$$

where $F(r)$ is the mass function which is always positive in the interval $[0, r_b]$, with $a(r)$ being another function of integration. Clearly, R is positive in the above interval, so $t \neq a(r)$ as long as R and F are positive. Computing the derivative of R with respect to r , we get, $R' = [3F'(t - a)^2]/(4R^2) - (3/2)(F/R^2)(t - a(r))a'(r)$. Using the above expression for $t - a(r)$ we get,

$$R' = \frac{1}{3} \frac{RF'}{F} + a'(r) \frac{\sqrt{F}}{\sqrt{R}} \quad (12)$$

Since F being the mass function is an increasing function of r , we have $F' > 0$ in the interval $[0, r_b]$. Using (r, v) coordinates, where v is a function of (r, t) and where we have put $R = rv(r, t)$, and $F(r) = r^3 M(r)$, we get,

$$v' = \frac{v}{3} \frac{M'}{M} + a'(r) \frac{\sqrt{M}}{\sqrt{v}} \quad (13)$$

We get a supremum for $|v'|$, and we use the above to get a $\delta > 0$ such that for $0 \leq r \leq \delta$, R' is strictly positive, thereby avoiding shell-crossings. Towards this end, noting that $\epsilon \leq v \leq 1$ gives $(1/\sqrt{v}) \leq (1/\sqrt{\epsilon})$. Since functions $M(r)$ and $a(r)$ are at least C^1 , and M is non-zero in the interval $[0, r_b]$, so the functions M'/M and $a'(r)\sqrt{M}$ are continuous on the compact interval $[0, r_b]$, and hence bounded. Thus, let $|M'/M| \leq K_1$ and $|a'(r)(\sqrt{M})| \leq K_2$ over the above interval. So we get $|v'| \leq K$ on $[0, r_b]$, where $K = (1/3)K_1 + (K_2)/\sqrt{\epsilon}$. This gives the desired δ where $\delta = \epsilon/K$ and we can carry out the proof of showing that $R' > 0$ as before.

We thus see that for a collapsing matter cloud, upto any arbitrarily small value $v = \epsilon$ corresponding to any very late stage in collapse, there is always a finite radius δ around the center throughout the collapse such that there are no shell-crossings in that ball. So if we are interested in examining the visibility or otherwise locally of any arbitrarily small collapsing ball just before the final central shell-focusing singularity, that can be done without bothering about the shell-crosses all the way, which do not exist in a δ comoving radius around the center of the collapsing cloud. Physically this is interesting, because as the collapse progresses, at a certain very late stage one would expect the classical relativity to break down and quantum gravity takes over. So if we call that stage of collapse to be $v = \epsilon$, a very small quantity, in reality we are interested in examining the visibility or otherwise of such an extreme late stage of collapse only, or of the quantum ball that formed in collapse where quantum gravity takes over at a very late stage.

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